The Ulam spiral unraveled.

Introduction.

Little is known about the exact distribution of the prime numbers, and yet prime numbers can exhibit stunning regularities. A fine example is the spiral of the Polish mathematician Stanislaw Ulam (Fig. 1). Ulam discovered on certain diagonals in the spiral patterns within the prime numbers that are still unexplained.

In this article the Ulam spiral is completely unraveled. By placing a counterclockwise Ulam spiral with start value zero (the Ulam 0–spiral) in a Cartesian coordinate system, patterns within the prime numbers can be examined both analytically and geometrically.
It is found that the Ulam 0–spiral is completely defined by eight families of quadratic functions. Factorable members within the eight families determine the elimination of natural numbers as prime number in the spiral.
The definition of the eight families of functions makes it possible to examine any rectangular area or diagonal, without fully constructing the spiral.
With the eight families of functions the Cartesian coordinates of a given natural number can be calculated in a few steps.

100	99	98	97	96	95	94	93	92	91
65	64	63	62	61	60	59	58	57	90
66	37	36	35	34	33	32	31	56	89
67	38	17	16	15	14	13	30	55	88
68	39	18	5	4	3	12	29	54	87
69	40	19	6	1	2	11	28	53	86
70	41	20	7	8	9	10	27	52	85
71	42	21	22	23	24	25	26	51	84
72	43	44	45	46	47	48	49	50	83
73	74	75	76	77	78	79	80	81	82

Fig. 1: The counterclockwise Ulam spiral.

Decennia old questions about the Ulam spiral.

The Ulam spiral, as discovered in 1963, is special because the graphical display shows that prime numbers have the tendency to appear on certain diagonals within the spiral. These clear patterns continue even when the spiral grows bigger. The spiral can start with the initial value 1 as used by Ulam (Fig. 2a), or with any other natural number.
A start value 41 gives an uninterrupted sequence of 40 prime numbers (Fig. 2b) that is reducible to Euler's famous formula for prime numbers $n^2 + n + 41$.

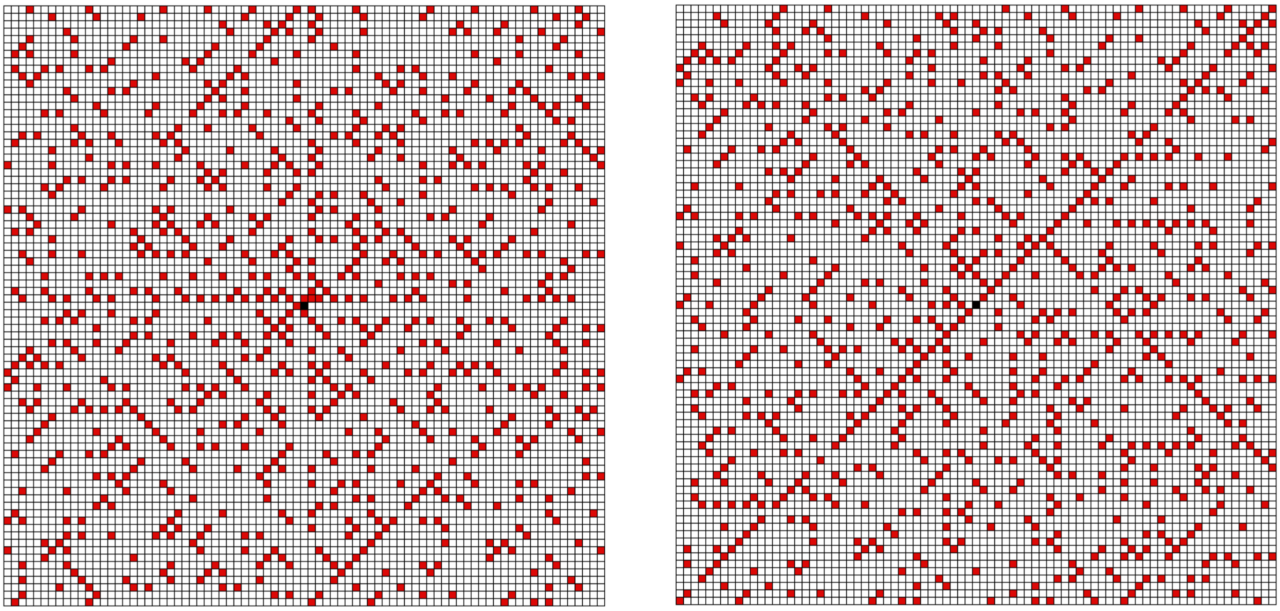

Fig. 2ab: An 81 x 81 matrix of the Ulam spiral with start value 1 (left) and 41 (right)

According to Ulam and his team the patterns in the spiral imply that there are many combinations $b, c \in \mathbf{Z}$, for which the function $f(n) = 4n^2 + bn + c$ generates more prime numbers than other combinations.
Ulam and his team came up with unanswered questions like:

1. Are prime numbers equally distributed over each quadrant of the grid?
2. Are there lines that contain infinitely many prime numbers?

The Ulam spiral in a Cartesian coordinate system.

In this paper a counterclockwise Ulam 0−spiral with start value 0 (Fig. 3b) is placed in the centre of a Cartesian coordinate system.

The spiral with startvalue 41 in Fig. 2b has the function $f(n) = 4n^2 + 2n + 41$ on the SW main diagonal. In the Ulam 0−spiral this diagonal visually appears as the 41th SW diagonal (Fig. 3c). The sequence $\{1847, 2021, 2203, 2393, \ldots\}$ belongs via $n \longmapsto n + 21$ also to the function $f(n) = 4n^2 + 170n + 1847$. Ulam and his team found for the latter function for numbers smaller than 10 million a ratio of 0.466 for the prime numbers.

In the spiral with startvalue 59 the SE main diagonal contains many prime numbers and has the matching function $f(n) = 4n^2 + 4n + 59$. When the spiral starts with the value 0, this diagonal visually appears as the 59th SE diagonal. The sequence $\{3539, 3779, 4027, 4283, \ldots\}$ (Fig. 3a) thus also belongs to the function $f(n) = 4n^2 + 236n + 3539$.

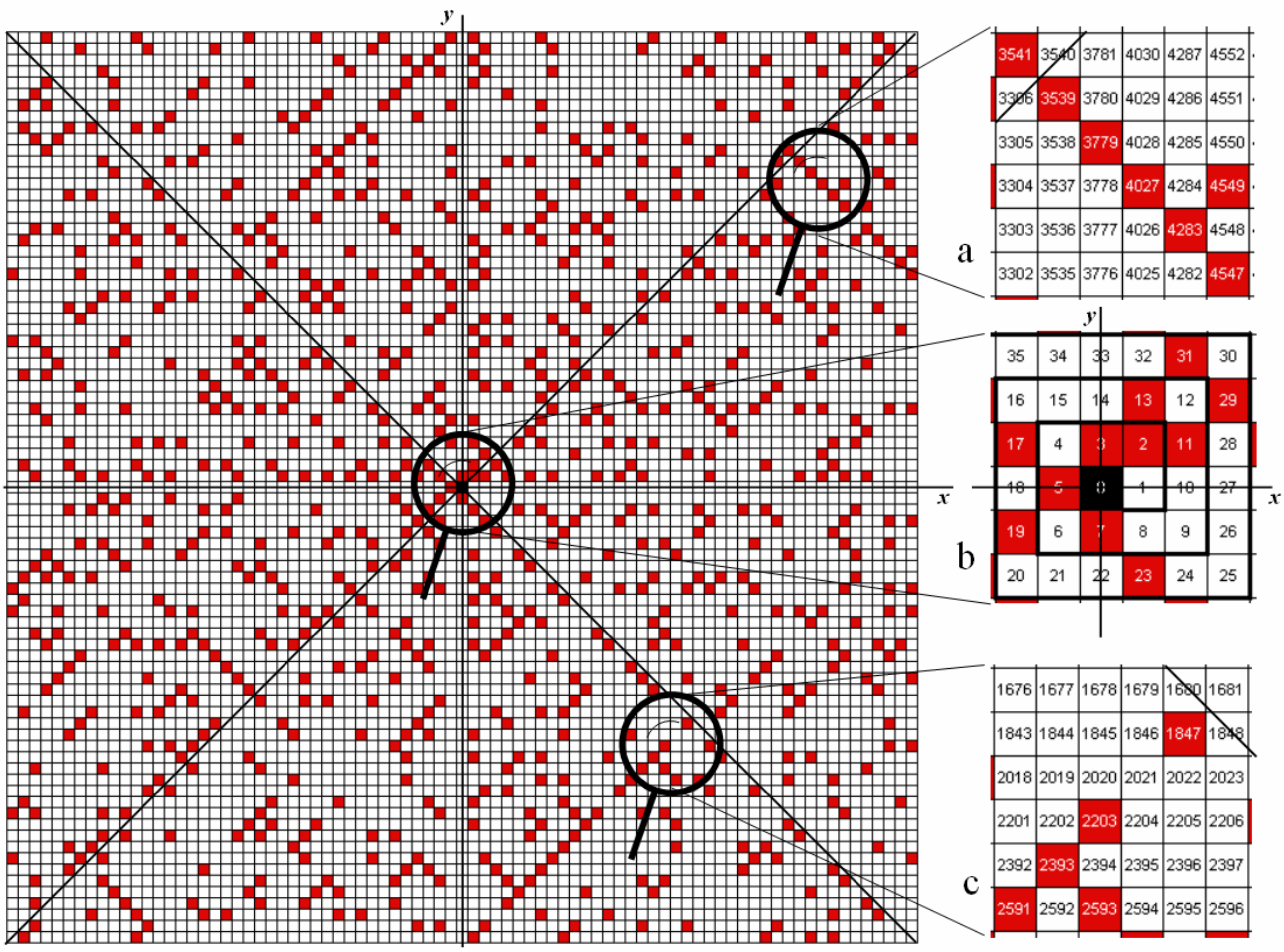

Fig. 3: An 81 x 81 matrix of the Ulam 0−spiral with startvalue 0.

Eight families of functions.

When the counterclockwise Ulam 0−spiral is placed in the centre of a Cartesian plane the spiral is fully defined by the eight families of functions $f_{b,c}(n) = 4n^2 + bn + c$, with $n \in \mathbf{N}_0$, $b, c \in \mathbf{Z}$ and $-3 \le b \le 4$.
Using the compass rose the value b is linked to the wind directions, e.g. the quadratic functions of NE diagonals with $b = -2$ can also be written as $f_c(n_{NE}) = 4n^2 - 2n + c$ (Fig. 4).
The linear projections of members of the eight families of functions start at their last intersection with the line $|y| = |x|$.
The coordinates of this intersection defines the function.

Each number > 0 in the Ulam 0−spiral belongs to three consecutive families of functions, see for example the functions at the points S, T and U in Fig. 4. Point V on the NE main diagonal is member of five families of functions. Further calculations (see "coordinates of a natural number") defines point V in the E sector, and therefore as member of the NE, E and SE family.

With the eight families of functions any rectangular area or diagonal of the Ulam 0−spiral can be generated, without fully constructing the spiral.

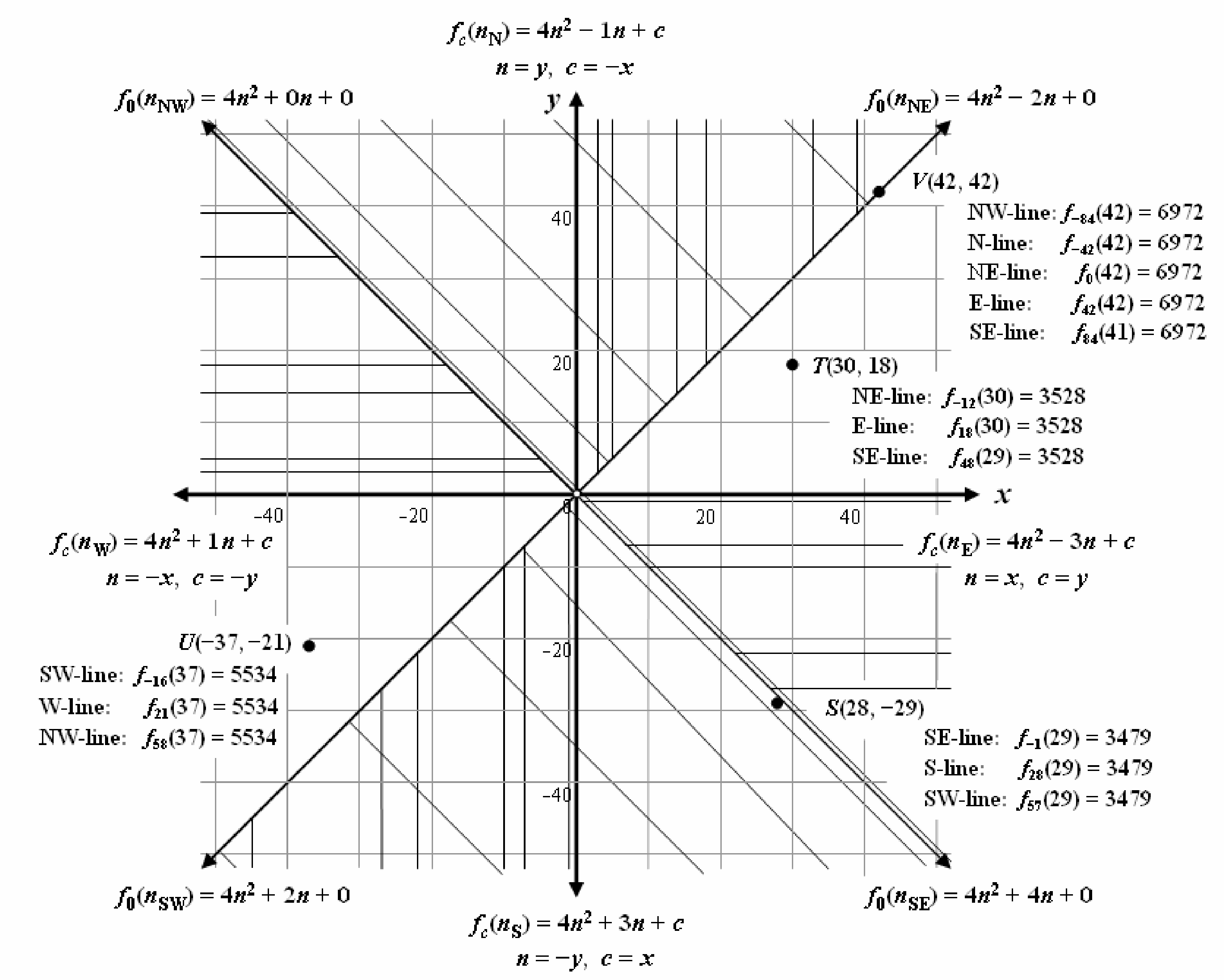

Fig. 4: The eight families of functions and their special factorable members.

Factorable functions within the Ulam 0−spiral.

The eight families of functions from the Ulam 0−spiral contain regular factorable members who have no prime numbers > p_4, i.e. the family members with $c = 0$ and all diagonals with $c \equiv 0 \pmod 2$.
Special factorable members from the eight families of functions in the Ulam 0−spiral are responsable for quickly eliminating natural numbers as prime number on other odd-diagonals, since if $d \mid f(n)$ then $d \mid f(n + d)$

The E, N, W and S functions, and the NW and SE functions with $c \equiv 1 \pmod 2$, have infinitely many values $c(k)$ whereby a special family member can be factored, see the table below.
The NE and SW functions with $c \equiv 1 \pmod 2$ can never be resolved. Fig. 4 shows the linear projections of all special members within the selected area, from their last intersection with the line $|y| = |x|$ onwards.
The SE diagonal is defined as $f_c(n_{SE}) = 4n^2 + 4n + c$ and becomes via $n \mapsto n - 1$ the function $f(n) = 4n^2 - 4n + c$ with almost identical features. There is a discontinuity at the SE main diagonal, see also further down.

Direction	b	Family of functions (with $c \in \mathbf{Z}$)	Special members (with $k \in \mathbf{Z}$)	$c(k)$ −values of special factorable members
E	−3	$f_c(n_E) = 4n^2 - 3n + c$	$c = -k\,(4k - 3)$	{ 0, −1, −7, −10, −22, −27, −45, … }
NE	−2	$f_c(n_{NE}) = 4n^2 - 2n + c$		
N	−1	$f_c(n_N) = 4n^2 - 1n + c$	$c = -k\,(4k - 1)$	{ 0, −3, −5, −14, −18, −33, −39, … }
NW	0	$f_c(n_{NW}) = 4n^2 + 0n + c$	$c = -(2k - 1)^2$	{ −1, −9, −25, −49, −81, −121, … }
W	1	$f_c(n_W) = 4n^2 + 1n + c$	$c = -k\,(4k + 1)$	{ 0, −3, −5, −14, −18, −33, −39, … }
SW	2	$f_c(n_{SW}) = 4n^2 + 2n + c$		
S	3	$f_c(n_S) = 4n^2 + 3n + c$	$c = -k\,(4k + 3)$	{ 0, −1, −7, −10, −22, −27, −45, … }
SE	4	$f_c(n_{SE}) = 4n^2 + 4n + c$	$c = -4k^2 + 1$	{ 1, −3, −15, −35, −63, −99, … }
	(−4)	$f(n) = 4n^2 - 4n + c$	$c = -4k^2 + 1$	{ 1, −3, −15, −35, −63, −99, … }

Coordinates of a natural number in the spiral.

For any natural number g in the Ulam 0−spiral the coordinates in the Cartesian coordinate system can be calculated through the families of functions. Define $m = \sqrt{g / 4}$ with $m \in \mathbf{R}_{\geq 0}$ and $n = \lfloor m \rceil$ with $n \in \mathbf{N_0}$.
The value $m - n$ determines the sector in which g lies, see the table below.

$m - n$	Sector	Function	(x, y)
$-½ \leq m - n < -¼$	E	$f_c(n_E) = 4n^2 - 3n + c$	(n, c)
$-¼ \leq m - n < 0$	N	$f_c(n_N) = 4n^2 - 1n + c$	$(-c, n)$
$0 \leq m - n < ¼$	W	$f_c(n_W) = 4n^2 + 1n + c$	$(-n, -c)$
$¼ \leq m - n < ½$	S	$f_c(n_S) = 4n^2 + 3n + c$	$(c, -n)$

For example in the next table the coordinates are recalculated for the points S, T, U and V from Fig. 4.

g	m	n	Function	c	(x, y)
3479	29,491…	29	$f_c(n_S) = 4n^2 + 3n + c$	28	$S\,(28, -29)$
3528	29,698…	30	$f_c(n_E) = 4n^2 - 3n + c$	18	$T\,(30, 18)$
5534	37,195…	37	$f_c(n_W) = 4n^2 + 1n + c$	21	$U\,(-37, -21)$
6972	41,749…	42	$f_c(n_E) = 4n^2 - 3n + c$	42	$V\,(42, 42)$

The distribution of prime numbers within the Ulam 0−spiral

By placing the counterclockwise Ulam 0−spiral in a Cartesian coordinate system the quadratic functions of the eight families become visible via their projections. Patterns in the prime numbers can thus be examined both analytically and geometrically.
Geometrically, the intersection can be determined of the projections of the quadratic function with special factorable members in the eight families of functions. When an intersection coincides with a lattice point the element is composite. The corresponding divisors than affect the behavior of elements within the family member via a fixed pattern.

The function $f_{41}(n_{SW})$ in a Cartesian coordinate system

The function $f_{41}(n_{SW}) = 4n^2 + 2n + 41$ from the Ulam 0−spiral appears in the Cartesian coordinate system at $n \geq 21$ as the SW diagonal $f(n) = 4n^2 + 170n + 1847$ and the corresponding linear projection $x = y + 41$. For smaller values of n the projection of $f_{41}(n_{SW}) = 4n^2 + 2n + 41$ deflects with every crossing of the line $|y| = |x|$ closer to the origin via a generic pattern, see the linear functions in Fig. 5. The discontinuity on the SE main diagonal generates a translation over $(1, -1)$.

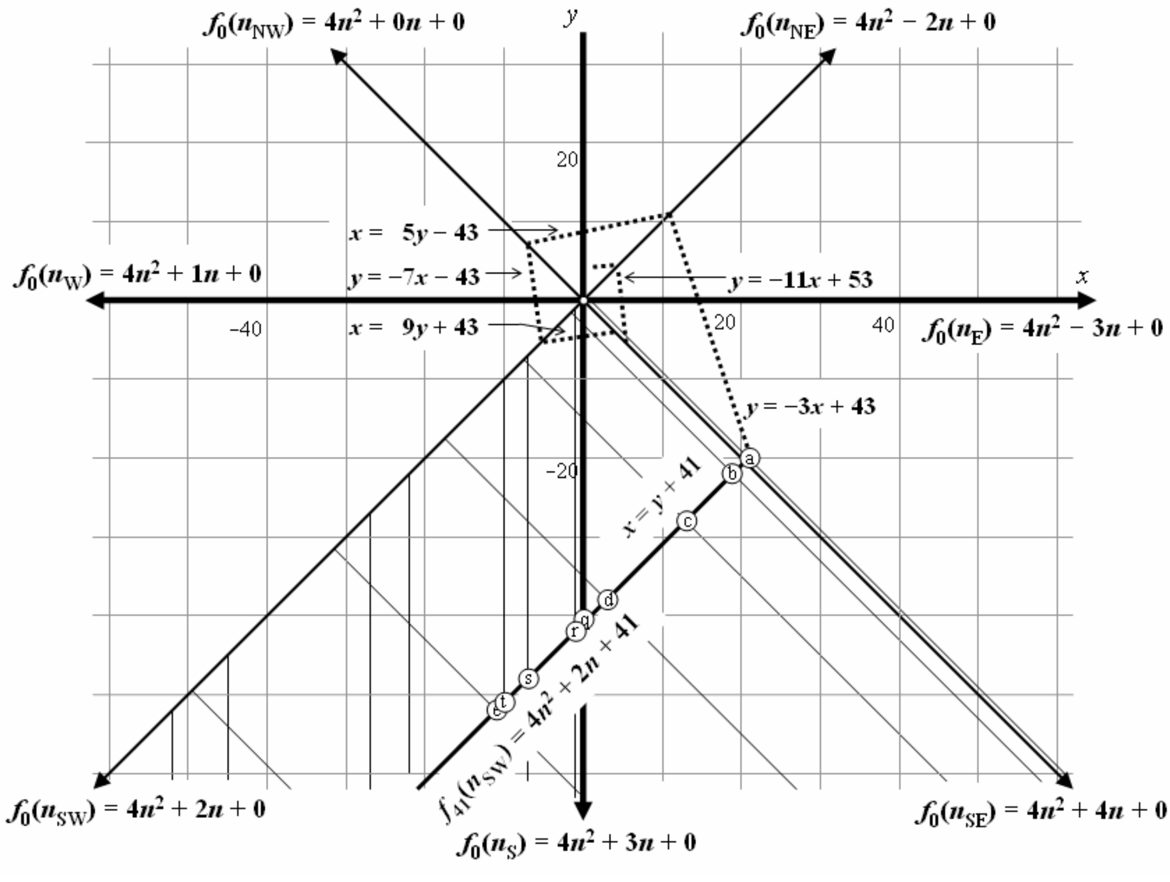

Fig. 5: The function $f_{41}(n_{SW})$ in a Cartesian coordinate system.

In the Cartesian coordinate system the projection of the function $f_{41}(n_{\mathrm{SW}}) = 4n^2 + 2n + 41$ has infinitely many lattice points that coincide with special factorable SE and S functions that contain no prime numbers $> p_4$ (Fig. 5).
The composite numbers g in $f_{41}(n_{\mathrm{SW}}) = 4n^2 + 2n + 41$ show patterns like:

- SE: $g \in$ {a, b, c, d, … $\mid g = f_{41}(n_{\mathrm{SW}}) \bullet f_{41}(-n_{\mathrm{SW}}) \lor g = f_{41}(n_{\mathrm{SW}}) \bullet f_{41}(n_{\mathrm{NE}})$ }
- S: $g \in$ {q, s, u, w, … $\mid g = (4 \bullet f_{41}(n_{\mathrm{SE}}) + 3) \bullet f_{41}(n_{\mathrm{SW}})$ }
- S: $g \in$ {q, r, t, v, … $\mid g = (4 \bullet f_{41}(-n_{\mathrm{SE}}) + 3) \bullet f_{41}(-n_{\mathrm{SW}})$ }

A function value $f(n)$ is composite if $f(n) = d_A \bullet d_B$ with $d_x \mid f(n)$, $gcd\,(f(n), d_x) > 1 \;\; \forall\, d_x \in \{d_A, d_B\}$
If d_A is a divisor, then $d_A \mid f(n + d_A \bullet m)$ and $d_B(m) = f(n + d_A \bullet m) \,/\, d_A$ with $m \in \mathbf{N}_0$. Also if d_B is a divisor, then $d_B \mid f(n + d_B \bullet m)$ and $d_A(m) = f(n + d_B \bullet m) \,/\, d_B$. When $d_A(m) = d_B(m)$ the divisors generate new composite function values via a single pattern instead of a double pattern, see point a in the table below.

#	$f_{41}(n_{\mathrm{SW}}) = 4n^2 + 2n + 41$	Coordinate	Pattern divisor A	Pattern divisor B
a	$f_{41}(20) = 1681 = 41 \bullet 41$	A (21, –20)	$n = 20 + 41 \bullet m$	$n = 20 + 41 \bullet m$
b	$f_{41}(22) = 2021 = 47 \bullet 43$	B (19, –22)	$n = 22 + 47 \bullet m$	$n = 22 + 43 \bullet m$
c	$f_{41}(28) = 3233 = 61 \bullet 53$	C (13, –28)	$n = 28 + 61 \bullet m$	$n = 28 + 53 \bullet m$
d	$f_{41}(38) = 5893 = 83 \bullet 71$	D (3, –38)	$n = 38 + 83 \bullet m$	$n = 38 + 71 \bullet m$

#	$f_{41}(n_{\mathrm{SW}}) = 4n^2 + 2n + 41$	Coordinate	Pattern divisor A	Pattern divisor B
q	$f_{41}(41) = 6847 = 167 \bullet 41$	Q (0, –41)	$n = 41 + 167 \bullet m$	$n = 41 + 41 \bullet m$
r	$f_{41}(42) = 7181 = 167 \bullet 43$	R (–1, –42)	$n = 42 + 167 \bullet m$	$n = 42 + 43 \bullet m$
s	$f_{41}(48) = 9353 = 199 \bullet 47$	S (–7, –48)	$n = 48 + 199 \bullet m$	$n = 48 + 47 \bullet m$
t	$f_{41}(51) = 10547 = 199 \bullet 53$	T (–10, –51)	$n = 51 + 199 \bullet m$	$n = 51 + 53 \bullet m$
u	$f_{41}(63) = 16043 = 263 \bullet 61$	U (–22, –63)	$n = 63 + 263 \bullet m$	$n = 63 + 61 \bullet m$
v	$f_{41}(68) = 18673 = 263 \bullet 71$	V (–27, –68)	$n = 68 + 263 \bullet m$	$n = 68 + 71 \bullet m$
w	$f_{41}(86) = 29797 = 359 \bullet 83$	W (–45, –86)	$n = 86 + 359 \bullet m$	$n = 86 + 83 \bullet m$

Pattern divisor B of $f_{41}(n_{\mathrm{SW}})$ contains further regularities based on the coordinates and the divisors, see below.

#	Pattern divisor B	$d_B(m) \mid f_{41}(n_{\mathrm{SW}})$ with $m \in \mathbf{N_0}$	Sequence $d_B(m)$
a	$n = 20 + 41 \bullet m$	$41 \bullet 4m^2 + (20 \bullet 8 + 2)\, m + 41$	{41, 367, 1021, … }
b	$n = 22 + 43 \bullet m$	$43 \bullet 4m^2 + (22 \bullet 8 + 2)\, m + 47$	{47, 397, 1091, … }
c	$n = 28 + 53 \bullet m$	$53 \bullet 4m^2 + (28 \bullet 8 + 2)\, m + 61$	{61, 499, 1361, … }
d	$n = 38 + 71 \bullet m$	$71 \bullet 4m^2 + (38 \bullet 8 + 2)\, m + 83$	{83, 673, 1831, … }

#	Pattern divisor B	$d_B(m) \mid f_{41}(n_{\mathrm{SW}})$ with $m \in \mathbf{N_0}$	Sequence $d_B(m)$
q	$n = 41 + 41 \bullet m$	$41 \bullet 4m^2 + (41 \bullet 8 + 2)\, m + 167$	{167, 661, 1483, … }
	$n = (41 - 41) + 41 \bullet m$	$41 \bullet 4m^2 + ((41 - 41) \bullet 8 + 2)\, m + 1$	$m \longmapsto m + 1$
s	$n = 48 + 47 \bullet m$	$47 \bullet 4m^2 + (48 \bullet 8 + 2)\, m + 199$	{199, 773, 1723, … }
	$n = (48 - 47) + 47 \bullet m$	$47 \bullet 4m^2 + ((48 - 47) \bullet 8 + 2)\, m + 1$	$m \longmapsto m + 1$
u	$n = 63 + 61 \bullet m$	$61 \bullet 4m^2 + (63 \bullet 8 + 2)\, m + 263$	{263, 1013, 2251, … }
	$n = (63 - 61) + 61 \bullet m$	$61 \bullet 4m^2 + ((63 - 61) \bullet 8 + 2)\, m + 1$	$m \longmapsto m + 1$
w	$n = 86 + 83 \bullet m$	$83 \bullet 4m^2 + (86 \bullet 8 + 2)\, m + 359$	{359, 1381, 3067, … }
	$n = (86 - 83) + 83 \bullet m$	$83 \bullet 4m^2 + ((86 - 83) \bullet 8 + 2)\, m + 1$	$m \longmapsto m + 1$

For the (x, y) coordinates of the composite numbers $g \in$ { q, r, s, t, … } further applies:

- $g \in$ { q, r, s, t, … $\mid x = c = -k\,(4k - 3)$ from $f_c(n_{\mathrm{E}})$ with $k \in \mathbf{Z}$ } OR
  $g \in$ { q, r, s, t, … $\mid x = c = -k\,(4k + 3)$ from $f_c(n_{\mathrm{S}})$ with $k \in \mathbf{Z}$ }
- $g \in$ { q, s, u, w, … $\mid y = -f_{41}(n_{\mathrm{S}}) = -f_{41}(-n_{\mathrm{E}})$ with $n_{\mathrm{S}}, n_{\mathrm{E}} \in \mathbf{N_0}$ and $n_{\mathrm{S}} = n_{\mathrm{E}}$ }
- $g \in$ { q, r, t, v, … $\mid y = -f_{41}(-n_{\mathrm{S}}) = -f_{41}(n_{\mathrm{E}})$ with $n_{\mathrm{S}}, n_{\mathrm{E}} \in \mathbf{N_0}$ and $n_{\mathrm{S}} = n_{\mathrm{E}}$ }

The function $f_{59}(n_{SE})$ in a Cartesian coordinate system.

The function $f_{59}(n_{SE}) = 4n^2 + 4n + 59$ from the Ulam 0−spiral appears in het Cartesian coordinate system at $n \geq 29$ as the SE diagonal $f(n) = 4n^2 + 236n + 3539$ and the corresponding linear projection $y = -x + 59$. For smaller values of n the projection of $f_{59}(n_{SE}) = 4n^2 + 4n + 59$ deflects with every crossing of the line $|y| = |x|$ closer to the origin via a generic pattern, see the linear functions in Fig. 6. The discontinuity on the SE main diagonal gives a translation over $(1, -1)$.

In the Cartesian coordinate system the projection of the function $f_{59}(n_{SE}) = 4n^2 + 4n + 59$ has infinitely many lattice points that coincide with special factorable NW and E functions that contain no prime numbers $> p_4$ (Fig. 6).
The composite numbers g from $f_{59}(n_{SE}) = 4n^2 + 4n + 59$ show patterns like:

- NW: $g \in \{a, b, c, d, e, \dots \mid g = (\tfrac{1}{2} \cdot f_{59}(n_{SE})) \cdot (\tfrac{1}{2} \cdot f_{59}(-n_{SE}))$ with $n_{SE} \mapsto n_{SE} + \tfrac{1}{2}\}$
- E: $g \in \{q, r, s, t, u, \dots \mid g = (4 \cdot f_{59}(n_{SW}) - 3) \cdot f_{59}(n_{SE}) \vee g = (4 \cdot f_{59}(-n_{SW} - 1) - 3) \cdot f_{59}(n_{SE})\}$

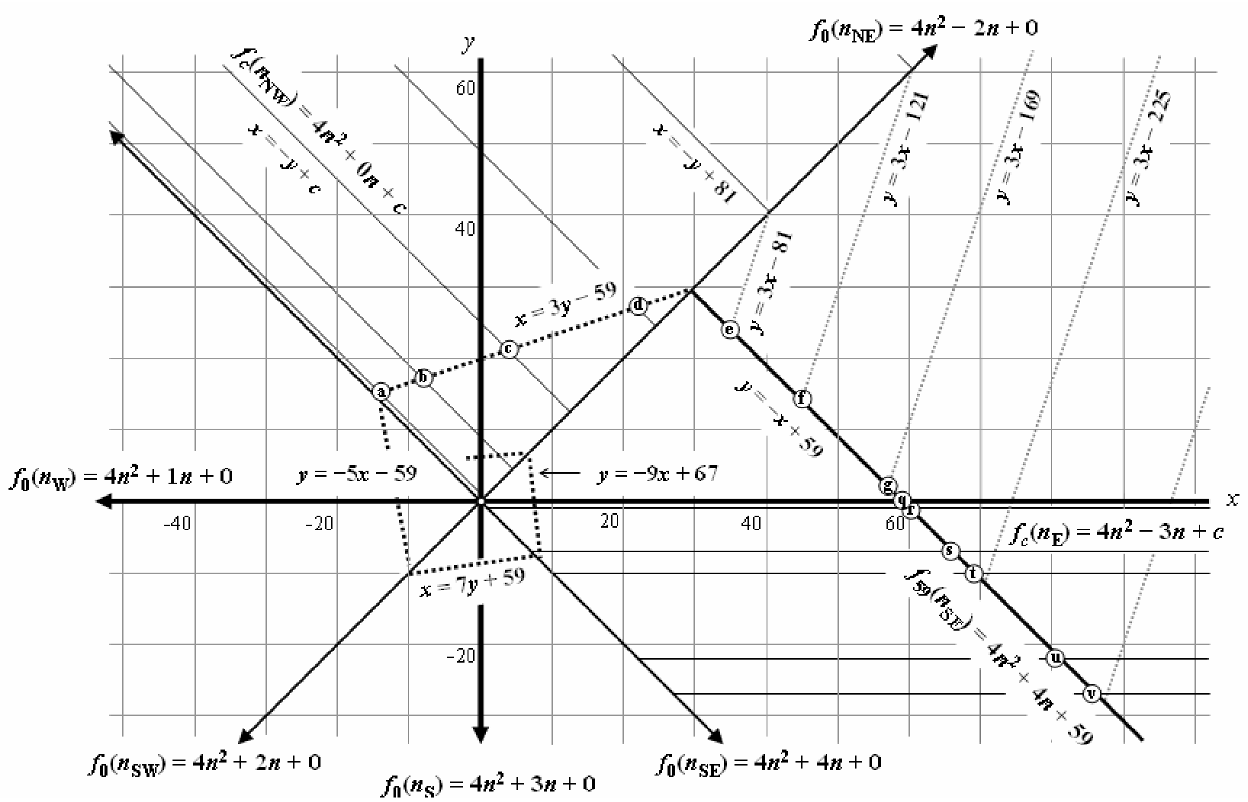

Fig. 6: The function $f_{59}(n_{SE})$ in a Cartesian coordinate system.

The density of prime numbers on SE diagonals.

With the eight families of functions of the Ulam 0−spiral any rectangular area or diagonal can by examined. Fig. 7 shows the density of prime numbers in $f_c(n_{SE}) = 4n^2 + 4n + c$ up to $f_c(n_{SE}) = 10^9$ with $n \in \mathbf{N}_0$ and $-20 \le c \le 60$. Formally SE diagonals start at their last intersection with the line $|\,y\,| = |\,x\,|$ and thus at a higher n.
SE diagonals with $c \equiv 0 \pmod 2$ and SE diagonals with $c \in \{1, -3, -15, \ldots \mid c = -4k^2 + 1$ with $k \in \mathbf{Z}\}$ contain no prime numbers $> p_4$. A high density of prime numbers is found in $f_{59}(n_{SE}) = 4n^2 + 4n + 59$.
Other SE diagonals with a high concentration of prime numbers are for example at $c \in \{-397, -361, 233, 653\}$.

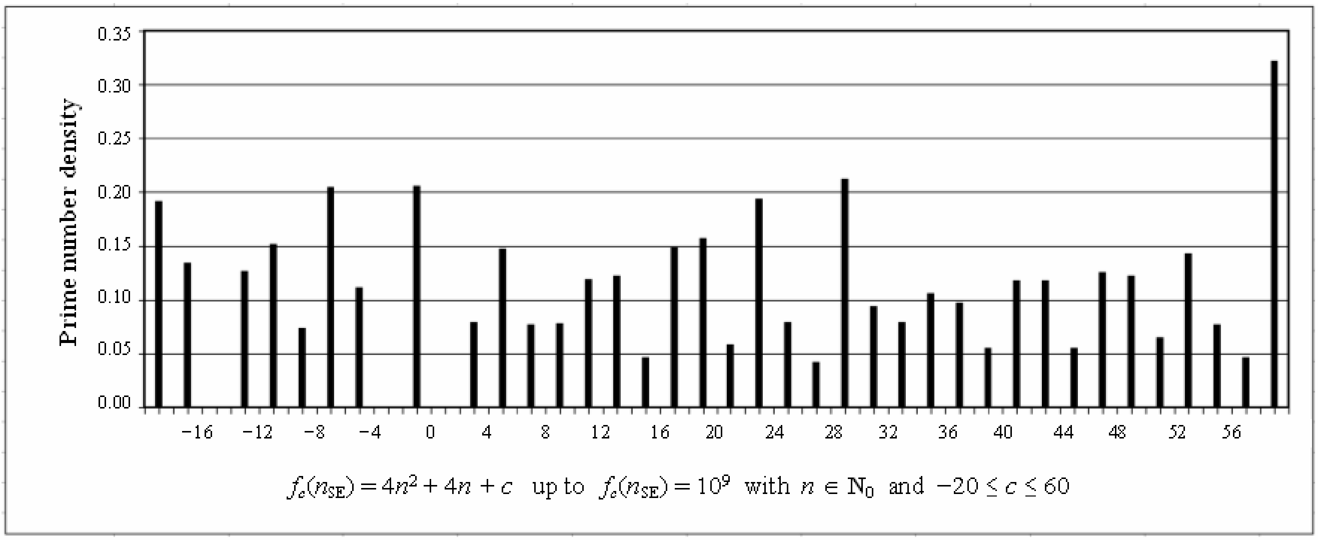

Fig. 7: Prime number density on SE diagonals.

Results of the Ulam 0−spiral in a Cartesian coordinate system.

By placing the counterclockwise Ulam 0−spiral in a Cartesian coordinate system concrete research, both analytically and geometrically, is possible into the distribution of prime numbers. Any rectangular area or diagonal can be examined without fully constructing the spiral.

In the Ulam 0−spiral all natural numbers are completely defined by the eight families of functions $f_{b,c}(n) = 4n^2 + bn + c$, with $n \in \mathbf{N}_0$, $b, c \in \mathbf{Z}$ and $-3 \le b \le 4$.

For any given natural number in the Ulam spiral the coordinates in the Cartesian coordinate system can be calculated via the families of functions in a few steps.

This research shows that the Ulam 0−spiral contains many regularities that are obeyed with stunning precision. Despite the clear and explainable patterns on the diagonals in the Ulam spiral no new algorithms (as yet) have been found to generate large amounts of prime numbers.

Ulam's open questions.

1. Are prime numbers equally distributed over each quadrant of the grid?

While studying the Ulam 0−spiral in a Cartesian coordinate system no apparent differences were found in the distribution of prime numbers over the quadrants of the grid or in the N, E, S and W sectors.

2. Are there lines that contain infinitely many prime numbers?

Fig. 8 shows for different functions the ratio $\boldsymbol{r}\,(f\,(n))$ of prime numbers in $f(n)$ up to $f(n) = 10^9$.
The $\boldsymbol{r}\,(f\,(n) = n) = 0.05$ at $f(n) = 10^9$ corresponds with $\boldsymbol{\pi}(\mathrm{n}) / \mathrm{n} \approx 1 / (\log(\mathrm{n}) - 1)$ from the number theory.

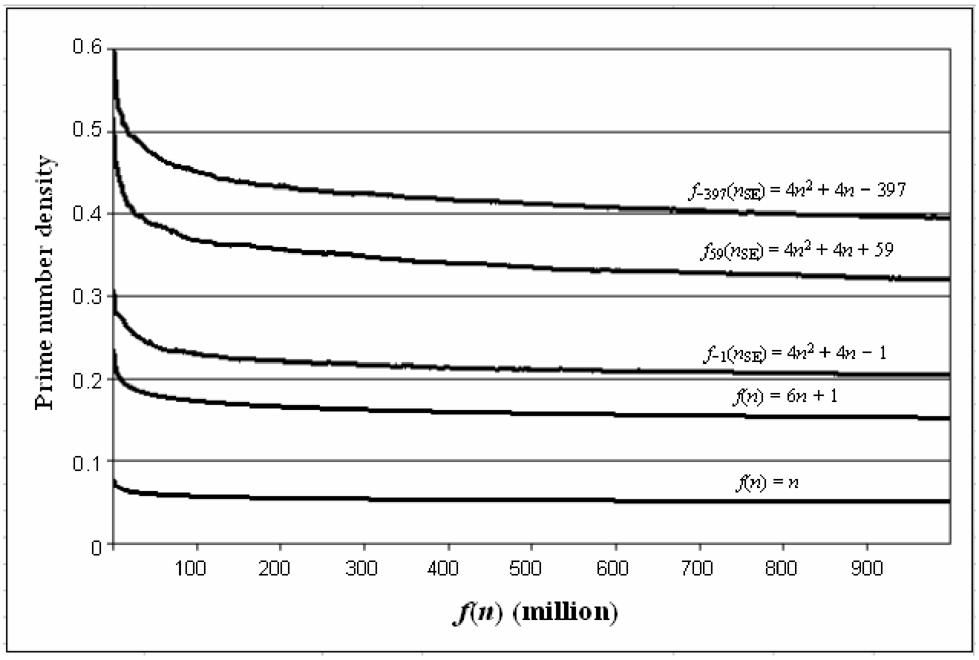

Fig. 8: Prime numbers density in different functions.

The $\boldsymbol{r}\,(f_{b,c}(n) = 4n^2 + bn + c)$ with $-3 \le b \le 4$ approaches $C_{b,c} \bullet \boldsymbol{r}\,(f\,(n) = n)$ by equal function values, with $C_{b,c}$ a constant in $\mathbf{R}_{\ge 0}$. The prime number filter function $f(n) = 6n \pm 1$ also satisfies this pattern with $C = 3.00$.
Within the Ulam 0−spiral $C_{b,c} = 0$ for factorable members of the eight families of functions.
In the table below some functions are summarized, amongst others those from Fig. 8.

Function	Translation	Diagonal in the 0−spiral	r $(f(n))$	$C_{b,c}$
$f_{-1}(n_{SE}) = 4n^2 + 4n - 1$	$n \longmapsto n + 1$	$f(n) = 4n^2 + 12n + 7$	0.20	$C_{4,-1} = 4.0$
$f_{59}(n_{SE}) = 4n^2 + 4n + 59$	$n \longmapsto n + 29$	$f(n) = 4n^2 + 236n + 3539$	0.32	$C_{4,59} = 6.3$
$f_{41}(n_{SW}) = 4n^2 + 2n + 41$	$n \longmapsto n + 21$	$f(n) = 4n^2 + 170n + 1847$	0.36	$C_{2,41} = 7.1$
$f_{-397}(n_{SE}) = 4n^2 + 4n - 397$	$n \longmapsto n + 199$	$f(n) = 4n^2 + 1596n + 158803$	0.39	$C_{4,-397} = 7.8$

This study presents the conjecture that in the Ulam 0−spiral diagonals with prime numbers > p_4 contain infinitely many prime numbers.

References.

[1] Gardner, M. (1971). *Martin Gardner's Sixth Book of Mathematical Diversions from Scientific American*, University of Chicago Press
[2] Stein, M. L., Ulam, S. M & Wells, M. B. (1964). A visual display of some properties of the distribution of primes. *American Mathematical Monthly* 71:516–520.
[3] Wells, David (2011), *Prime Numbers: The Most Mysterious Figures in Math*, John Wiley & Sons