Prime number pattern in the Ulam spiral.

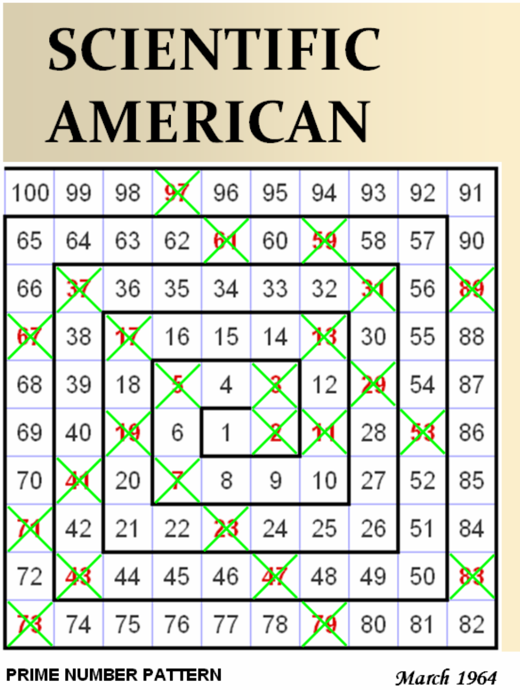

Fig. 1. the Ulam spiral on the cover of Scientific American.

Prime numbers are considered the 'building blocks' of mathematics, because every natural number is either a prime or a unique product of primes. Mathematicians are fascinated by the infinitely many prime numbers, because there seems to be no formula to generate all prime numbers.

The Polish-American mathematician Stanislaw Ulam used a grid of horizontal and vertical lines to visualize the distribution of prime numbers. Starting in the middle he numbered the intersections counterclockwise, following a square spiral pattern. He marked the primes and found prominent diagonal, horizontal, and vertical lines containing large numbers of primes. The pattern stands out even more in a larger grid.
The spiral featured on the cover of the March 1964 issue of Scientific American (Fig. 1).

Fig. 2 shows a 101 x 101 counterclockwise Ulam spiral where each black dot represents a prime number. The black dots appear to create diagonals in every direction. There are also diagonals with empty spaces where prime numbers are notably missing.
Diagonal, horizontal, and vertical lines in the number spiral correspond to polynomials of the form $f(x) = 4x^2 + bx + c$, with b and c integer constants.
When b is even, the lines are diagonal, and either all numbers are odd, or all are even, depending on the value of c.

The prime number sequence {5, 19, 41, 71} on the SW diagonal in Fig. 1 belongs to the function $f(x) = 4x^2 + 10x + 5$. Similarly, the {7, 23, 47, 79} sequence of prime numbers along the SE diagonal matches the function $f(x) = 4x^2 + 12x + 7$.
The patterns suggests that there are many integer constants b and c where the function $f(x) = 4x^2 + bx + c$ comes up with more primes then on similar lines.

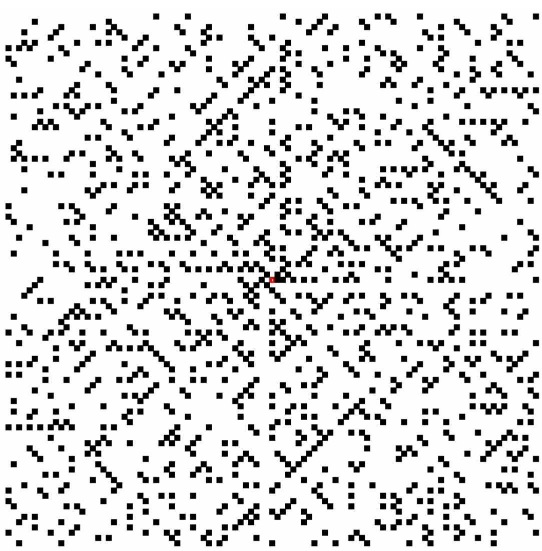

Fig. 2. a 101 x 101 counterclockwise Ulam spiral

The counterclockwise Ulam spiral can begin with any natural number.
Starting with 41 the spiral produces a grid with an amazing, unbroken sequence of 40 primes along the NE and SW main diagonal. It corresponds with Euler's most famous prime generator $x^2 + x + 41$ (Fig. 3a). Checking all such numbers less than 10 million, Ulam and his coworkers found the proportion of primes to be 0.475.
The researchers also identified several other formulas rich with primes. For numbers of the form $4x^2 + 170x + 1847$, the proportion of primes is 0.466; for $4x^2 + 4x + 59$ (Fig. 3b) this ratio is 0.437.

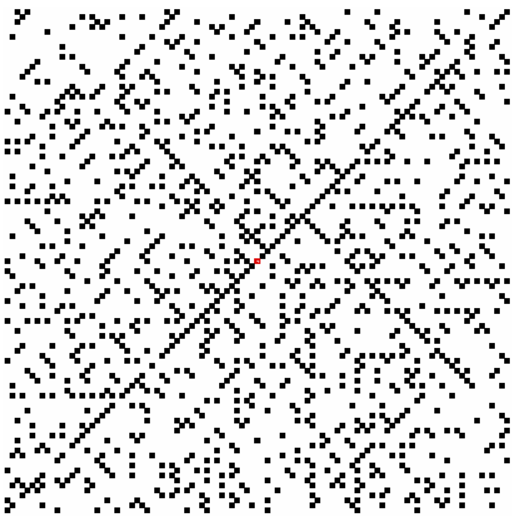

Fig. 3a. an Ulam spiral with startvalue 41.

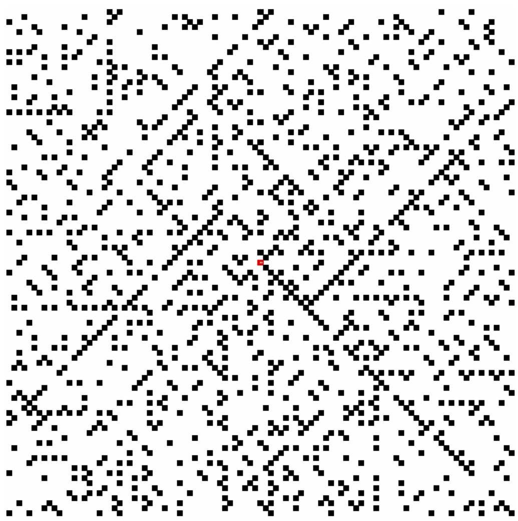

Fig. 3b. an Ulam spiral with startvalue 59.

Studying the Ulam spiral.

Not much is known about the Ulam spiral, because it is not heavily analysed by mathematicians. Still the Ulam spiral may be important since it shows a clear pattern among prime numbers. These patterns might give enough information to discover new polynomials.
In this study a counterclockwise Ulam spiral with startvalue 0 is placed in a Cartesian coordinate system.
Beginning with 0 at (0, 0) the outwards spiraled coordinates are:
1: (1, 0) 2: (1, 1) 3: (0, 1) 4: (−1, 1) etc

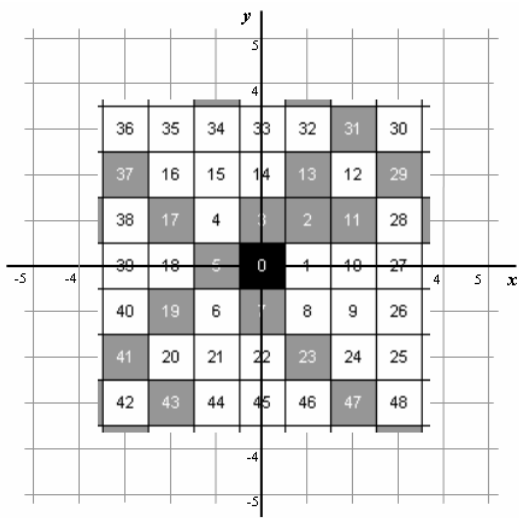

Fig. 4. Implementing the Cartesian coordinate system.

The Ulam spiral unraveled.

When the counterclockwise Ulam 0−spiral is placed in the centre of a Cartesian plane, the spiral is fully defined by the eight families of functions $f_{b,c}(n) = 4n^2 + bn + c$, with $n \in \mathbf{N}_0$, $b, c \in \mathbf{Z}$ and $-3 \leq b \leq 4$ (Fig. 5).
Using the compass rose the value b is linked to the wind directions, e.g. the quadratic functions of SE diagonals, with $b = 4$, can also be written as $f_c(n_{\text{SE}}) = 4n^2 + 4n + c$.
The linear projections of members of the eight families of functions appear as diagonal, horizontal or vertical lines onwards from their last crossing of the line $|y| = |x|$.
Also depicted in Fig. 5 are special factorable members of six families of functions that have no prime numbers $> p_4$.
The odd NE and SW functions can never be resolved.

Each number in the Ulam 0−spiral appears on the straight lines of three consecutive families of functions, see for example the functions at the points S, T and U in Fig. 5. Point V on the NE main diagonal is member of five families of functions. Calculations for obtaining the coordinates place point V in the E sector, and thus only as member of the NE, E and SE family.

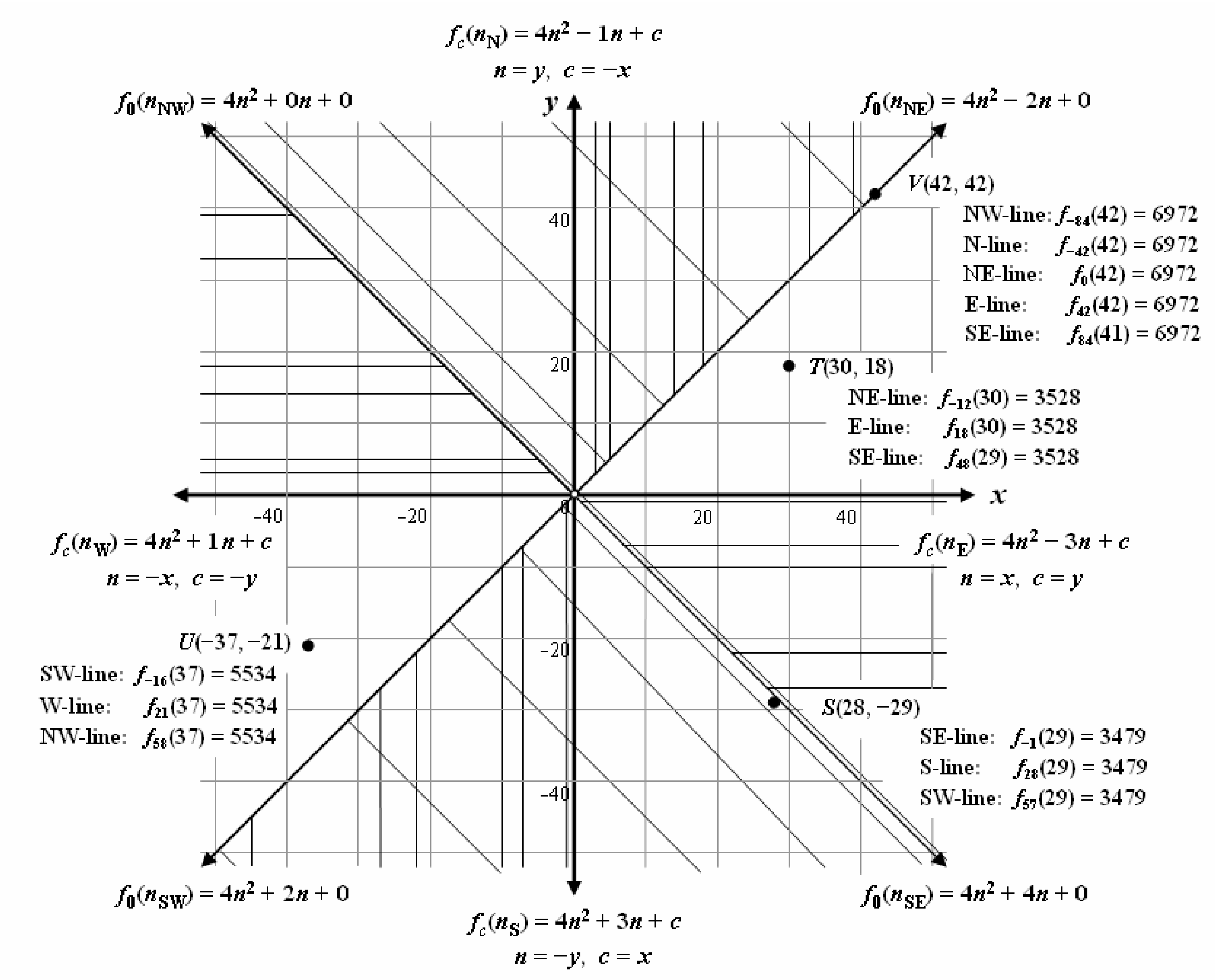

Fig. 5. definition of the eight families of functions that describe the counterclockwise Ulam 0−spiral.

The Ulam spiral conundrum.

An explanation for the visual clustering of prime numbers on certain diagonals can be found by observing the density of prime numbers on the SE diagonals. Fig. 6 shows the ratio of prime numbers in the functions $f_c(n_{SE}) = 4n^2 + 4n + c$ up to $f_c(n_{SE}) = 10^9$ with $n \in \mathbf{N_0}$ and $-20 \leq c \leq 60$.
The function $f_{59}(n_{SE})$ with many prime numbers stands out because it is next to the function $f_{57}(n_{SE})$, which has significant less prime numbers. The function $f_{-1}(n_{SE})$ has a relative high ratio of prime numbers, while flanked by the special factorable functions $f_{-3}(n_{SE})$ and $f_1(n_{SE})$ that contain no prime numbers $> p_4$.

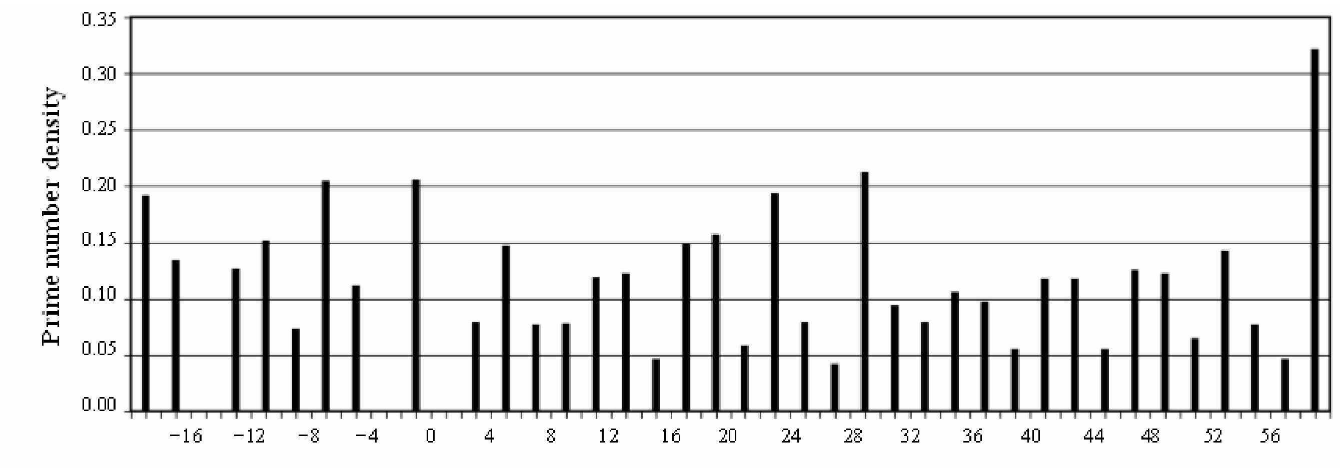

Fig. 6. prime number density for $f_c(n_{SE}) = 4n^2 + 4n + c$ up to $f_c(n_{SE}) = 10^9$ with $n \in \mathbf{N_0}$ and $-20 \leq c \leq 60$.

The influence of the special factorable functions.

When in a counterclockwise Ulam spiral the path of a function meets the path of a special factorable function on a lattice point, the natural number on the lattice point is composite. Six of the eight families of functions have these special factorable members.

Fig. 7 shows the function $f_{59}(n_{SE}) = 4n^2 + 4n + 59$ from the Ulam 0–spiral in het Cartesian coordinate system.
For $n \geq 29$ the SE diagonal appears as the linear projection $y = -x + 59$. For smaller values of n the projection of $f_{59}(n_{SE})$ deflects closer to the origin with every crossing of the line $|y| = |x|$.
The first composite number is $f_{59}(14) = 899 = d_A \bullet d_B = 29 \bullet 31$ due to $f_{-1}(n_{NW}) = 4n^2 + 0n - 1$.
The divisor d_A eliminates further natural numbers, since $d_A \mid f(n + d_A \bullet m)$ with $m \in \mathbf{N}_0$.

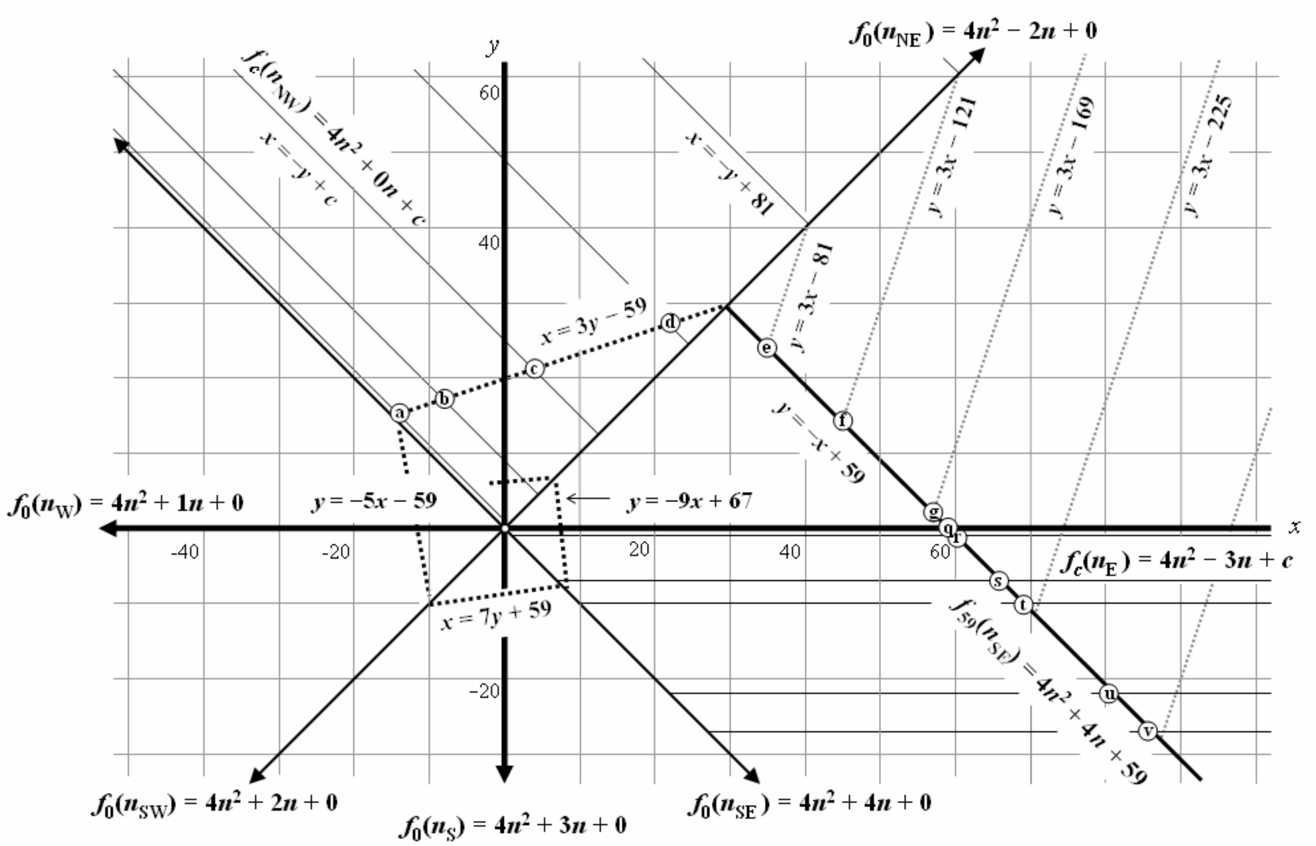

Fig. 7. the influence of special factorable functions on the function $f_{59}(n_{SE})$

Pratical use of the eight families of functions.

The definition of the eight families of functions makes it possible to study the counterclockwise Ulam 0−spiral in detail, without fully constructing the spiral. It also supplies answers to Ulam's open questions:

1. There is no significant difference found in the distribution of prime numbers over the four quadrants in the Cartesian coordinate system.
2. It is the conjecture that the prime number density in $f_{b,c}(n) = 4n^2 + bn + c$ approaches $C_{b,\,c}$ times the density of the prime numbers in $f(n) = n$. The special factorable functions and the even diagonals have $C_{b,\,c} = 0$.
   The SE diagonal $f_{-397}(n_{SE}) = 4n^2 + 4n - 397$, rich with prime numbers, has $C_{4,\,-397} = 7.8$. Other well known dense lines are $f_{41}(n_{SW}) = 4n^2 + 2n + 41$ with $C_{2,\,41} = 7.1$ and $f_{59}(n_{SE}) = 4n^2 + 4n + 59$ with $C_{4,\,59} = 6.3$.

References.

[1] Gardner, M. (1971). *Martin Gardner's Sixth Book of Mathematical Diversions from Scientific American*, University of Chicago Press

[2] Stein, M. L., Ulam, S. M & Wells, M. B. (1964). A visual display of some properties of the distribution of primes. *American Mathematical Monthly* 71:516–520.

[3] Wells, David (2011), *Prime Numbers: The Most Mysterious Figures in Math*, John Wiley & Sons

[4] Website: http://www.primorial-sieve.com/The Ulam spiral unraveled